

# FIRST NON-ABELIAN COHOMOLOGY OF TOPOLOGICAL GROUPS II

H. SAHLEH \* AND H.E. KOSHKOSHI

**ABSTRACT.** In this paper we introduce a new definition of the first non-abelian cohomology of topological groups. We relate the cohomology of a normal subgroup  $N$  of a topological group  $G$  and the quotient  $G/N$  to the cohomology of  $G$ . We get the inflation-restriction exact sequence. Also, we obtain a seven-term exact cohomology sequence up to dimension 2. We give an interpretation of the first non-abelian cohomology of a topological group by the notion of a principle homogeneous space.

## 1. INTRODUCTION

The first non-abelian cohomology of group  $G$  with coefficients in a crossed  $G$ -module  $(A, \mu)$  was (algebraically) introduced by Guin [1, 2]. The Guin's approach extended by H. Inassaridze to any dimension of non-abelian cohomology of  $G$  with coefficients in a (partially) crossed topological  $G - R$ -bimodule  $(A, \mu)$  (see [4, 5, 6]). We generalize the Inassaridze's approach to define the first cohomology of non-abelian cohomology of topological groups. We continue to study non-abelian cohomology of topological groups (see [7, 8]).

In this paper, topological groups are not necessarily abelian, unless otherwise specified. Let  $G$  and  $A$  be topological groups. It is said that  $A$  is a topological  $G$ -module, whenever  $G$  acts continuously on the left of  $A$ . For all  $g \in G$  and  $a \in A$  we denote the action of  $g$  on  $a$  by  ${}^g a$ . If  $A$  is a topological  $G$ -module,  $H^1(G, A)$  denotes the first cohomology of  $G$  with coefficients in  $A$  [8]. The group isomorphism is denoted by  $\cong$ . The center and the commutator subgroup of a topological group  $G$  is denoted by  $Z(G)$  and  $[G, G]$ , respectively. If the topological groups  $G$  and  $R$  act continuously on a topological group  $A$ , then the notation  ${}^{gr}a$  means  ${}^g({}^r a)$ ,  $g \in G$ ,  $r \in R$ ,  $a \in A$ . We assume that every topological group acts on itself by conjugation. The coboundary map  $\delta_G^n$  is as in [3,

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\*Corresponding author .

Definition (3.1)]. If  $G$  and  $H$  are topological groups, then  $\mathbf{1} : G \rightarrow H$  denotes the trivial homomorphism. If  $f : G \rightarrow H$  is a map, then in what follows  $f^{-1} : G \rightarrow H$  denotes a map given by  $f^{-1}(g) = f(g)^{-1}$ . Note that if  $P$  is a principle homogeneous space over a partially crossed topological  $G - R$ -bimodule  $(A, \mu)$ , then  $e_p^{-1} : P \rightarrow A$  is the inverse map of the homeomorphism  $e_p : A \rightarrow P$ ,  $p \in P$  (see section 5).

In section 2, using the notion of partially crossed topological  $G - R$ -bimodule  $(A, \mu)$  we define  $Der_c(G, (A, \mu))$  as in [7] and the first non-abelian cohomology,  $H^1(G, (A, \mu))$ , of  $G$  with coefficients in  $(A, \mu)$  as a quotient of  $Der_c(G, (A, \mu))$  (see Definition 2.9). Every topological  $G$ -module  $A$  realises a (partially) crossed topological  $G - A/Z(A)$ -bimodule  $(A, \pi_A)$ . So, we may define the first cohomology,  $\bar{H}^1(G, A)$ , of  $G$  with coefficients in  $A$  as follows:

$$\bar{H}^1(G, A) = H^1(G, (A, \pi_A))$$

In general this definition of the first non-abelian cohomology differs from the first pointed set cohomology,  $H^1(G, A)$ , which is defined in [8]. We show that there is a natural injection  $\zeta : H^1(G, (A, \mu)) \rightarrow H^1(G, A)$ . Thus, there is an injection  $\bar{H}^1(G, A) \rightarrow H^1(G, A)$ . As a result, we may identify  $\bar{H}^1(G, A)$  with  $H^1(G, A)$ , whenever  $H^1(G, A/Z(A))$  is null. In particular, if  $A$  is abelian, then  $\bar{H}^1(G, A) \cong H^1(G, A)$ .

In section 3, we introduce a new concept called *cocompatible triple* to study the change of groups. Using the cocompatible triple we define the inflation and the restriction maps and will show that for any normal subgroup  $N$  of  $G$ , there is an exact sequence

$$1 \longrightarrow H^1(G/N, (A^N, \mu^N)) \xrightarrow{Inf^1} H^1(G, (A, \mu)) \xrightarrow{Res^1} H^1(N, (A, \mu))^{G/N}$$

In section 4, we show that for every proper extension,  $1 \rightarrow (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \rightarrow 1$ , with continuous sections there exists a seven-term exact sequence,

$$0 \rightarrow H^0(G, A) \xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\iota^1} H^1(G, (B, \mu)) \xrightarrow{\pi^1} H^1(G, (C, \lambda)) \xrightarrow{\delta^1} H^2(G, A).$$

In section 5, we define a principle homogeneous space (or topological torsor) over a partially crossed topological  $G - R$ -bimodule  $(A, \mu)$  and we show that the first non-abelian cohomology of  $G$  with coefficients in  $(A, \mu)$  classifies the set of all principle homogeneous spaces over  $(A, \mu)$ . We denote by  $\mathcal{P}(A, \mu)$  the set of all classes of principle homogeneous spaces over  $(A, \mu)$ . Thus, naturally there exists a group product on  $\mathcal{P}(A, \mu)$ , whenever  $(A, \mu)$  satisfying conditions of Theorem 2.23. As a

result, if  $(A, \mu)$  is a topological crossed  $G$ -module, then  $\mathcal{P}(A, \mu)$  is a group (not necessarily abelian).

## 2. PARTIALLY CROSSED TOPOLOGICAL BIMODULES AND THE FIRST NON-ABELIAN COHOMOLOGY.

In this section, we introduce a partially crossed topological  $G - R$ -bimodule  $(A, \mu)$  to define the first non-abelian cohomology,  $H^1(G, (A, \mu))$ , of  $G$  with coefficients in  $(A, \mu)$ .

**Definition 2.1.** A precrossed topological  $R$ -module  $(A, \mu)$  consists of a topological  $R$ -module  $A$  and a continuous homomorphism  $\mu : A \rightarrow R$  such that

$$\mu({}^r a) = {}^r \mu(a), \quad r \in R, a \in A.$$

If in addition we have

$$\mu({}^a b) = {}^a b, \quad a, b \in A,$$

then  $(A, \mu)$  is called a crossed topological  $R$ -module.

**Definition 2.2.** A precrossed topological  $R$ -module  $(A, \mu)$  is said to be a partially crossed topological  $R$ -module, whenever it satisfies the following equality

$$\mu({}^a b) = {}^a b,$$

for all  $b \in A$  and for all  $a \in A$  such that  $\mu(a) \in [R, R]$ .

It is clear that every crossed topological  $R$ -module is a partially crossed topological  $R$ -module.

**Definition 2.3.** Let  $G$ ,  $R$  and  $A$  be topological groups. Precrossed topological  $R$ -module  $(A, \mu)$  is said to be a precrossed topological  $G - R$ -bimodule, whenever

- (1)  $G$  continuously acts on  $R$  and  $A$ ;
- (2)  $\mu : A \rightarrow R$  is a  $G$ -homomorphism;
- (3)  $({}^{gr})a = {}^{grg^{-1}}a$  for all  $g \in G$ ,  $r \in R$  and  $a \in A$ .

**Definition 2.4.** A precrossed topological  $G - R$ -bimodule  $(A, \mu)$  is said to be a (partially) crossed topological  $G - R$ -bimodule, if  $(A, \mu)$  is a (partially) crossed topological  $R$ -module.

**Example 2.5.** (1) Let  $A$  be an arbitrary topological  $G$ -module. Obviously,  $Z(A)$  is a topological  $G$ -module. Now, we define an action of  $R = A/Z(A)$  on  $A$  and an action of  $G$  on  $R$  by:

$${}^{aZ(A)}b = {}^a b, \quad \forall a, b \in A, \quad {}^g(aZ(A)) = {}^g aZ(A), \quad \forall g \in G, a \in A. \quad (2.1)$$

Let  $\pi_A : A \rightarrow R$  be the canonical homomorphism. It is easy to see that under (2.1) the pair  $(A, \pi_A)$  is a crossed topological  $G - R$ -bimodule.

- (2) By part (1), for any topological group  $G$  the pair  $(G, \pi_G)$  is a crossed topological  $G - G/Z(G)$ -bimodule.

*Notice 2.6.* It is obvious that any precrossed (crossed or partially crossed) topological  $R$ -module is naturally viewed as a precrossed (crossed or partially crossed) topological  $R - R$ -bimodule.

**Definition 2.7.** A morphism  $f : (A, \mu) \rightarrow (B, \nu)$  of precrossed (crossed) topological  $G - R$ -bimodule is a continuous homomorphism of topological groups  $f : A \rightarrow B$  such that

- (1)  $f({}^r a) = {}^r f(a)$ ,  $r \in R$ ,  $a \in A$ ;
- (2)  $f({}^g a) = {}^g f(a)$ ,  $g \in G$ ,  $a \in A$ ;
- (3)  $\mu = \nu \circ f$ .

**Definition 2.8.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. Denote by  $Der(G, (A, \mu))$  the set of all pairs  $(\alpha, r)$  where  $\alpha$  is a crossed homomorphism from  $G$  into  $A$ , i.e.,  $\alpha(gh) = \alpha(g){}^g \alpha(h)$ ,  $\forall g, h \in G$ ; and  $r$  is an element of  $R$  such that

$$\mu \circ \alpha(g) = {}^r g r^{-1}, \forall g \in G.$$

Let  $Der_c(G, (A, \mu))$  be a subset of  $Der(G, (A, \mu))$  which is defined as follows:

$$Der_c(G, (A, \mu)) = \{(\alpha, r) \in Der(G, (A, \mu)) \mid \alpha \text{ is continuous}\}.$$

There is a group product  $\star$  in  $Der(G, (A, \mu))$  by  $(\alpha, r) \star (\beta, s) = (\alpha \star \beta, rs)$ , where  $\alpha \star \beta(g) = {}^r \beta(g) \alpha(g)$ ,  $\forall g \in G$  (see [4, Proposition 1.5]). It is easy to see that  $Der_c(G, (A, \mu))$  is a subgroup of  $Der(G, (A, \mu))$ .

Let  $R$  be a topological  $G$ -module, then we define

$$H^0(G, R) = \{r \mid {}^g r = r, \forall g \in G\}.$$

Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. H. Inassaridze [4] introduced the equivalence relation  $\sim$  on the group  $Der(G, (A, \mu))$  as  $(\alpha, r) \sim (\beta, s)$  whenever

$$\exists a \in A \wedge (\forall g \in G \Rightarrow \beta(g) = a^{-1} \alpha(g) {}^g a) \quad (2.2)$$

and

$$s = \mu(a)^{-1} r \text{ mod } H^0(G, R) \quad (2.3)$$

Let  $\sim'$  be the restriction of  $\sim$  to  $Der_c(G, (A, \mu))$ . Therefore,  $\sim'$  is an equivalence relation. In other word,  $(\alpha, r) \sim' (\beta, s)$  if and only if  $(\alpha, r) \sim (\beta, s)$ , whenever  $(\alpha, r), (\beta, s) \in Der_c(G, (A, \mu))$ .

Let  $A$  be a topological  $G$ -module. We denote by  $\text{Inn}(G, A)$  the set of all inner crossed homomorphisms from  $G$  into  $A$ , in other words:

$$\text{Inn}(G, A) = \{\text{Inn}(a) | a \in A, \text{Inn}(a)(g) = a^g a^{-1}\}$$

Similarly, if  $(A, \mu)$  is a partially crossed topological  $G - R$ -bimodule, then put

$$\text{Inn}(G, (A, \mu)) = \{(\text{Inn}(a), \mu(a)z) | a \in A, z \in H^0(G, R)\}.$$

**Definition 2.9.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. The quotient set  $\text{Der}_c(G, (A, \mu)) / \sim'$  will be called the first cohomology of  $G$  with the coefficients in  $(A, \mu)$  and is denoted by  $H^1(G, (A, \mu))$ .

**Fact 2.10.** Let  $(A, \mu)$  is a partially crossed topological  $G - R$ -bimodule. Then

- (1)  ${}^z g a = {}^g z a, \forall a \in A, g \in G, z \in H^0(G, R)$ .
- (2)  $\alpha(g) {}^{gr} a = {}^r g a \alpha(g), \forall g \in G, a \in A, (\alpha, r) \in \text{Der}_c(G, (A, \mu))$ .

*Proof.* See [4, p. 499] and [5, p. 315].  $\square$

**Definition 2.11.** Let  $A$  be a topological  $G$ -module. Denote by  $\bar{H}^1(G, A)$  the first cohomology of  $G$  with coefficients in  $A$  and define it as follows:

$$\bar{H}^1(G, A) = H^1(G, (A, \pi_A)).$$

Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule, then there is the following natural map

$$\begin{aligned} \zeta : H^1(G, (A, \mu)) &\rightarrow H^1(G, A) \\ [(\alpha, r)] &\mapsto [\alpha]. \end{aligned}$$

**Theorem 2.12.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. Then

- (i)  $H^1(G, (A, \mu))$  can be naturally embedded in  $H^1(G, A)$ .
- (ii) There is a bijection between  $H^1(G, (A, \mu))$  and  $H^1(G, A)$  if and only if the induced map  $\mu^1 : H^1(G, A) \rightarrow H^1(G, R)$  is trivial (that is,  $\mu^1 = \mathbf{1}$ ).

*Proof.* Let  $\zeta$  be the natural map mentioned above. (i). Suppose that  $\zeta[(\alpha, r)] = \zeta[(\beta, s)]$ . Hence there is  $a \in A$  such that  $\beta(g) = a^{-1} \alpha(g) {}^g a$ , for all  $g \in G$ . Hence,  $\mu \beta(g) = \mu(a)^{-1} \mu \alpha(g) {}^g \mu(a)$ . Thus,  $s {}^g s^{-1} = \mu(a)^{-1} r {}^g r^{-1} {}^g \mu(a)$ , for all  $g \in G$ . Therefore for all  $g \in G$ ,  ${}^g (r^{-1} \mu(a) s) = r^{-1} \mu(a) s$ , i.e.,  $r^{-1} \mu(a) s \in H^0(G, R)$ . This shows that  $\zeta$  is one to one.

(ii). Let  $\alpha \in \text{Der}_c(G, A)$ , then  $\mu \alpha \in \text{Der}_c(G, R)$ . If  $\mu^1 = \mathbf{1}$ , then there is an  $r \in R$  such that for all  $g \in G$ ,  $\mu \alpha(g) = r {}^g r^{-1}$ .

Thus  $\zeta([\alpha, r]) = [\alpha]$ . So,  $\zeta$  is onto. Conversely, if  $\zeta$  is onto. Let  $\alpha \in \text{Der}_c(G, A)$ , then there is  $(\beta, r) \in \text{Der}_c(G, (A, \mu))$  such that  $\zeta([\beta, r]) = [\alpha]$ . Thus, there is  $a \in A$  such that  $\alpha(g) = a^{-1}\beta(g)^ga$ , whence  $\mu^1([\alpha]) = \mu^1([\beta]) = [\mu\beta] = [\text{Inn}(r)] = 1$ , and this finishes the proof.  $\square$

**Corollary 2.13.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule and  $H^1(G, R) = 0$ . Then, there is a bijection between  $H^1(G, A)$  and  $H^1(G, (A, \mu))$ .*

*Proof.* The proof is immediate.  $\square$

Theorem 2.12 implies immediately the following two corollaries:

**Corollary 2.14.** *Let  $A$  be a topological  $G$ -module. There is a bijection between  $\bar{H}^1(G, A)$  and  $H^1(G, A)$  if and only if  $\pi_A^1 : H^1(G, A) \rightarrow H^1(G, A/Z(A))$  is null.*

**Corollary 2.15.** *If  $A$  is a topological  $G$ -module and  $H^1(G, A/Z(A)) = 0$ , then there is a bijection between  $\bar{H}^1(G, A)$  and  $H^1(G, A)$ . In particular if  $A$  is abelian, then  $\bar{H}^1(G, A) \cong H^1(G, A)$ .*

According to part (i) of the proof of Theorem 2.12, we have the next remark.

**Remark 2.16.** Let  $(A, \mu)$  be a partially crossed (topological)  $G - R$ -bimodule, and let  $(\alpha, r), (\beta, s) \in \text{Der}(G, (A, \mu))$ . Then

$$(\alpha, r) \sim (\beta, s) \Leftrightarrow \exists a \in A \wedge (\forall g \in G \Rightarrow \beta(g) = a^{-1}\alpha(g)^ga).$$

In other words, the condition (2.2) implies the condition (2.3). Thus, we can delete the condition (2.3) of the equivalence relation  $\sim$ .

**Definition 2.17.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. The quotient set  $\text{Der}_c(G, (A, \mu)) / \sim'$  will be called the first cohomology of  $G$  with the coefficients in  $(A, \mu)$  and is denoted by  $H^1(G, (A, \mu))$ .

**Theorem 2.18.** *Let  $(A, \mu)$  be a crossed topological  $G - R$ -bimodule. Then,  $\text{Inn}(G, (A, \mu))$  is a subgroup of  $\text{Der}_c(G, (A, \mu))$ .*

*Proof.* Suppose that  $(\text{Inn}(a), \mu(a)z), (\text{Inn}(a'), \mu(a')z') \in \text{Inn}(G, (A, \mu))$ . We get  $(\text{Inn}(a), \mu(a)z)(\text{Inn}(a'), \mu(a')z') = (\text{Inn}(a^za'), \mu(a^za')zz')$ , since for any  $g \in G$

$$\begin{aligned} \mu(a)^z \text{Inn}(a') \text{Inn}(a)(g) &= \mu(a)^z \text{Inn}(a')(g) \text{Inn}(a)(g) \\ &= a^z \text{Inn}(a')(g) a^{-1} \text{Inn}(a)(g) = a^z (a'^g a'^{-1}) a^{-1} (a^g a^{-1}) \\ &= a^z a'^g a'^{-1g} a^{-1} = a^z a'^g a'^{-1g} a^{-1} = a^z a'^g (a'^{-1} a^{-1}) \\ &= a^z a'^g (a^z a')^{-1} = \text{Inn}(a^z a')(g), \end{aligned}$$

and also,  $\mu(a)z\mu(a')z' = \mu(a)z\mu(a')z^{-1}zz' = \mu(a)^z\mu(a')zz' = \mu(a^z a')zz'$ . Thus,  $\text{Inn}(G, (A, \mu))$  is closed under multiplication.

In addition, We have  $(\text{Inn}(a), \mu(a)z)^{-1} = (\text{Inn}(z^{-1}a^{-1}), \mu(z^{-1}a^{-1})z^{-1})$ , since for any  $g \in G$

$$\begin{aligned} (\mu(a)z)^{-1}(\text{Inn}(a)(g))^{-1} &= z^{-1}\mu(a^{-1})(a^g a^{-1})^{-1} = z^{-1}\mu(a^{-1})(g a a^{-1}) = \\ &= z^{-1}(a^{-1g} a a^{-1} a) = z^{-1}(a^{-1g} a) = z^{-1}a^{-1}z^{-1}g a = z^{-1}a^{-1g}z^{-1}a = \\ &= z^{-1}a^{-1g}(z^{-1}a) = \text{Inn}(z^{-1}a^{-1})(g), \end{aligned}$$

and also,  $(\mu(a)z)^{-1} = z^{-1}\mu(a^{-1}) = \mu(z^{-1}a^{-1})z^{-1}$ . Therefore,  $\text{Inn}(G, (A, \mu))$  is closed under inversion. So,  $\text{Inn}(G, (A, \mu))$  is a subgroup of  $\text{Der}_c(G, (A, \mu))$ .  $\square$

*Remark 2.19.* Note that  $\text{Inn}(G, (A, \mu))$  is not necessarily a normal subgroup of  $\text{Der}_c(G, (A, \mu))$ . If  $H^1(G, (A, \mu))$  is a group, then  $\text{Inn}(G, (A, \mu))$  is a normal subgroup of  $\text{Der}_c(G, (A, \mu))$  and

$$H^1(G, (A, \mu)) \cong \text{Der}_c(G, (A, \mu)) / \text{Inn}(G, (A, \mu)).$$

*Remark 2.20.* Let  $(A, \mu)$  be a crossed topological  $G - R$ -bimodule. Then,  $\text{Inn}(G, (A, \mu))$  is a normal subgroup of  $\text{Der}_c(G, (A, \mu))$  if and only if  $({}^{rz})\alpha^{-1}\alpha \in \text{Inn}(G, A)$  for all  $(\alpha, r) \in \text{Der}_c(G, (A, \mu))$ ,  $z \in H^0(G, R)$ .

*Proof.* Suppose that  $(\alpha, r) \in \text{Der}_c(G, (A, \mu))$  and  $(\text{Inn}(a), \mu(a)z) \in \text{Inn}(G, (A, \mu))$ . We have

$$(\alpha, r)(\text{Inn}(a), \mu(a)z)(\alpha, r)^{-1} = (r\mu(a)zr^{-1}\alpha^{-1r}\text{Inn}(a)\alpha, r\mu(a)zr^{-1})$$

Thus,

$$\begin{aligned} (r\mu(a)zr^{-1}\alpha^{-1r}\text{Inn}(a)\alpha)(g) &= \mu^{(ra)}({}^{rz})\alpha^{-1}(g)^r \text{Inn}(a)(g)\alpha(g) \\ &= r a^{(rz)}\alpha^{-1}(g)^{rg} a^{-1}\alpha(g) = r a^{(rz)}\alpha^{-1}(g)\alpha(g)^{gr} a^{-1}. \end{aligned}$$

The last equality is obtained by part (2) of Fact 2.10. Hence, we have  $r\mu(a)zr^{-1}\alpha^{-1r}\text{Inn}(a)\alpha \sim' ({}^{rz})\alpha^{-1}\alpha$ . This completes the proof.  $\square$

As a consequence of Remark 2.20 we get the following corollary.

**Corollary 2.21.** *Let  $(A, \mu)$  be a crossed topological  $G - R$ -bimodule and  $A$  a trivial  $R$ -module. Then  $\text{Inn}(G, (A, \mu))$  is a normal subgroup of  $\text{Der}_c(G, (A, \mu))$ .*

Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. One can see there is a natural action of  $H^0(G, R)$  on  $H^1(G, (A, \mu))$  as follows:

$$z[(\alpha, r)] = [(z\alpha, {}^z r)], \quad z \in H^0(G, R), [(\alpha, r)] \in H^1(G, (A, \mu)),$$

where  $({}^z\alpha)(g) = {}^z\alpha(g)$ , for all  $g \in G$ . Note that by part (1) of Fact 2.10,  ${}^z\alpha$  is a crossed homomorphism.



**Lemma 2.22.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. If  $Der(G, (A, \mu))/\sim$  is a group, then  $H^1(G, (A, \mu))$  is isomorphic to a subgroup of  $Der(G, (A, \mu))/\sim$ .*

*Proof.* Clearly, the natural map  $j : H^1(G, (A, \mu)) \rightarrow Der(G, (A, \mu))/\sim$ ,  $[(\alpha, r)] \mapsto cls(\alpha, r)$  is injective. The equivalence relation  $\sim'$  is congruence, since by assumption,  $\sim$  is congruence. Thus,  $j$  is a homomorphism. This completes the proof.  $\square$

Indeed in Lemma 2.22,  $Der(G, (A, \mu))/\sim$  is the first non-abelian cohomology of  $G$  with coefficients in  $(A, \mu)$  [4, 6].

**Theorem 2.23.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule satisfying the following conditions*

- (i)  $H^0(G, R)$  is a normal subgroup of  $R$ ;
- (ii) for every  $c \in H^0(G, R)$  and  $(\alpha, r) \in Der_c(G, (A, \mu))$ , there exists  $a \in Ker\mu$  and  ${}^c\alpha(g) = a^{-1}\alpha(g)a$ ,  $\forall g \in G$ .

*Then,  $Der_c(G, (A, \mu))$  induces a group structure on  $H^1(G, (A, \mu))$ .*

*Proof.* By [4, Theorem 2.1], the quotient set  $Der(G, (A, \mu))/\sim$  is a group. Thus, by Lemma 2.22,  $H^1(G, (A, \mu))$  is a group.  $\square$

Let  $A$  and  $B$  be topological  $G$ -modules and let  $\mu : A \rightarrow B$  be a continuous  $G$ -homomorphism. We say that  $\mu$  is a  $G$ -retraction whenever there is a continuous  $G$ -homomorphism  $\rho : B \rightarrow A$  such that  $\mu\rho = Id_B$ . For example,  $(G, Id_G)$  is a crossed topological  $G$ -module and clearly,  $Id_G : G \rightarrow G$  is a  $G$ -retraction.

**Theorem 2.24.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule and suppose that  $\mu : A \rightarrow R$  is a  $G$ -retraction. Then, the following is an exact sequence.*

$$1 \rightarrow H^1(G, (A, \mu)) \xrightarrow{\zeta} H^1(G, A) \xrightarrow{\mu^1} H^1(G, R) \rightarrow 1$$

*Proof.* By part (i) of Theorem 2.12,  $\zeta$  is one to one. If  $(\alpha, r) \in Der_c(G, (A, \mu))$ , then  $\mu^1\zeta([\alpha, r]) = \mu^1([\alpha]) = [\mu \circ \alpha] = [Inn(r)] = 1$ . Thus  $Im\zeta \subset Ker\mu^1$ . Vice versa, if  $[\alpha] \in Ker\mu^1$ , then  $\mu\alpha$  is cohomologous to  $1$ . Hence, there is  $r \in R$  such that  $\mu\alpha(g) = r^g r^{-1}$ , for all  $g \in G$ . So,  $(\alpha, r) \in Der_c(G, (A, \mu))$  and  $\zeta([\alpha, r]) = [\alpha]$ . Therefore,  $Ker\mu^1 \subset Im\zeta$ . Finally, we show that  $\mu^1$  is onto. Suppose that  $\alpha \in Der_c(G, R)$ . Set  $\beta = \rho\alpha$ . Obviously,  $\beta \in Der_c(G, A)$  and  $\mu^1([\beta]) = [\alpha]$ .  $\square$

**Theorem 2.25.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule satisfying the following conditions:*

- (1)  $A$  and  $R$  are abelian;



- (2) for any  $r \in R$  and  $(\alpha, s) \in \text{Der}_c(G, (A, \mu))$  there exists  $a \in \text{Ker} \mu$  and  ${}^r\alpha(g) = a^{-1}\alpha(g)a$ ;
- (3)  $\mu : A \rightarrow R$  is a  $G$ -retraction.

Then,  $H^1(G, A) \cong H^1(G, (A, \mu)) \oplus H^1(G, R)$ .

*Proof.* The condition (1) implies that  $H^1(G, A)$  and  $H^1(G, R)$  are abelian groups. By Theorem 2.23,  $H^1(G, (A, \mu))$  is a group. The conditions (1) and (2) imply that the map  $\zeta$  is a homomorphism, since

$$\zeta([\alpha, r][\beta, s]) = \zeta([\alpha^r \beta], rs) = [\alpha^r \beta] = [\alpha][{}^r\beta] = [\alpha][\beta] = \zeta([\alpha, r])\zeta([\beta, s]).$$

Hence, by Theorem 2.24,  $1 \rightarrow H^1(G, (A, \mu)) \xrightarrow{\zeta} H^1(G, A) \xrightarrow{\mu^1} H^1(G, R) \rightarrow 1$  is an exact sequence (of groups and homomorphisms). By (3), there is a continuous  $G$ -homomorphism  $\rho$  such that  $\mu\rho = \text{Id}_R$ . Thus,  $\mu^1\rho^1 = (\mu\rho)^1 = (\text{Id}_R)^1 = \text{Id}_{H^1(G, R)}$ . This completes the proof.  $\square$

As an immediate result of Theorem 2.25, we have:

**Corollary 2.26.** *Let  $(A, \mu)$  be a partially crossed topological  $G$ -module satisfying the following conditions:*

- (1)  $G$  and  $A$  are abelian;
- (2)  $\mu : A \rightarrow G$  is a  $G$ -retraction.

Then,  $H^1(G, A) \cong H^1(G, (A, \mu)) \oplus \text{Hom}_c(G, G)$ .

Note that if  $A$  is an abelian topological  $G$ -module, then  $(\alpha, r) \sim' (\alpha, 1)$ , for every  $(\alpha, r) \in \text{Der}_c(G, (A, \mathbf{1}))$ . Therefore, we have the next theorem.

**Theorem 2.27.** *Let  $G$  be a topological group. Then,*

$$\tau_A : H^1(G, A) \rightarrow H^1(G, (A, \mathbf{1})), \quad \tau_A([\alpha]) = [(\alpha, 1)]$$

*is a natural isomorphism in the category of abelian topological  $G$ -modules.*

*Proof.* The proof is an standard argument.  $\square$

### 3. CHANGE OF GROUPS FOR THE FIRST COHOMOLOGY.

We introduce a notion called cocompatible triple and we get inflation-restriction exact sequence for the first non-abelian cohomology groups.

**Definition 3.1.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule and  $(A', \mu')$  a partially crossed topological  $G' - R'$ -bimodule. Suppose that  $\phi : G' \rightarrow G$ ,  $\varphi : R \rightarrow R'$  and  $\psi : A \rightarrow A'$  are continuous homomorphisms. The triple  $(\phi, \varphi, \psi)$  is called a cocompatible triple whenever the following conditions hold:

- (1)  ${}^{g'}\varphi(r) = \varphi(\phi(g')r)$ ,  $\forall g' \in G', r \in R$ ;

$$(2) \quad {}^{g'}\psi(a) = \psi(\phi({}^{g'}a)), \forall g' \in G', a \in A.$$

**Example 3.2.** If  $(A, \mu)$  is a partially crossed topological  $G$ – $R$ -bimodule and  $N$  a subgroup of  $G$ . Then,  $(A, \mu)$  is a partially crossed  $N$ – $R$ -bimodule. The triple  $(\iota, Id_R, Id_A)$  is a cocompatible triple, where  $\iota : N \rightarrow G$  is the inclusion map and  $Id_R$  and  $Id_A$  are the identity maps.

**Example 3.3.** If  $N$  is a normal subgroup of  $G$  and  $\mu^N : A^N \rightarrow R^N$  is the restriction of  $\mu : A \rightarrow R$ . Clearly  $(A^N, \mu^N)$  is a partially crossed topological  $G/N$ – $R^N$ -bimodule. The triple  $(\pi, \iota, j)$  is a cocompatible triple, where  $\pi : G \rightarrow G/N$  is the natural map,  $\iota : R^N \rightarrow R$  and  $j : A^N \rightarrow A$  are the inclusion maps.

Note that a cocompatible triple  $(\phi, \varphi, \psi)$  induces a natural map as follows:

$$Der_c(G, (A, \mu)) \rightarrow Der_c(G', (A', \mu')), (\alpha, r) \mapsto (\psi \circ \alpha \circ \phi, \varphi(r))$$

which induces naturally the map:

$$\begin{aligned} (\phi, \varphi, \psi)^1 : H^1(G, (A, \mu)) &\rightarrow H^1(G', (A', \mu')), \\ [(\alpha, r)] &\mapsto [(\psi \circ \alpha \circ \phi, \varphi(r))]. \end{aligned}$$

**Definition 3.4.** Let  $(A, \mu)$  be a partially crossed topological  $G$ – $R$ -bimodule and  $N$  a subgroup of  $G$ . The induced map  $(\iota, Id_R, Id_A)^1$  is called the restriction map and it is denoted by  $Res^1 : H^1(G, (A, \mu)) \rightarrow H^1(N, (A, \mu))$ .

**Definition 3.5.** Let  $(A, \mu)$  be a partially crossed topological  $G$ – $R$ -bimodule and  $N$  a normal subgroup of  $G$ . The induced map  $(\pi, \iota, j)^1$  is called the inflation map and it is denoted by  $Inf^1 : H^1(G/N, (A^N, \mu^N)) \rightarrow H^1(G, (A, \mu))$ .

**Lemma 3.6.** Let  $(A, \mu)$  be a partially crossed topological  $G$ – $R$ -bimodule, and  $N$  a normal subgroup of  $G$ . Then,

- (i)  $H^1(N, (A, \mu))$  is a  $G/N$ -set. Moreover, if  $H^1(N, (A, \mu))$  is a group, then  $H^1(N, (A, \mu))$  is a  $G/N$ -module.
- (ii)  $Im Res^1 \subset H^1(N, (A, \mu))^{G/N}$ .

*Proof.* (i) Since  $N$  is a normal subgroup of  $G$ , then, there is an action of  $G$  on  $Der_c(N, (A, \mu))$  as follows:

For every  $g \in G$  we define  ${}^g(\alpha, r) = (\tilde{\alpha}, {}^g r)$  with  $\tilde{\alpha}(n) = {}^g \alpha({}^{g^{-1}}n)$ ,  $n \in N$ .

In fact,  $\tilde{\alpha}$  is continuous and we have:

$$\begin{aligned} \tilde{\alpha}(mn) &= {}^g \alpha({}^{g^{-1}}(mn)) = {}^g \alpha({}^{g^{-1}}m {}^{g^{-1}}n) = {}^g \alpha({}^{g^{-1}}m) \cdot \\ &\quad {}^{mg} \alpha({}^{g^{-1}}n) = \tilde{\alpha}(m) {}^m \tilde{\alpha}(n), \end{aligned}$$

whence,  $\tilde{\alpha} \in \text{Der}_c(N, A)$ . Also it is easy to see that  $\mu(\tilde{\alpha}(n)) = (gr)^n(gr^{-1})$ , for every  $n \in N$ . Hence,  ${}^g(\alpha, r) \in \text{Der}_c(N, (A, \mu))$ . It is clear that  ${}^{gh}(\alpha, r) = {}^g({}^h(\alpha, r))$ . It is easy to verify that  ${}^g((\alpha, r)(\beta, s)) = {}^g(\alpha, r){}^g(\beta, s)$ . Consequently  $\text{Der}_c(N, (A, \mu))$  is a  $G$ -module. Now suppose that  $(\alpha, r) \sim (\beta, s)$ . Then, there is an  $a \in A$  such that  $\beta(n) = a^{-1}\alpha(n)^na, \forall n \in N$ . Thus, for every  $g \in G, n \in N$ ,

$${}^g\beta({}^{g^{-1}}n) = {}^ga^{-1}({}^g\alpha({}^{g^{-1}}n)){}^g({}^{g^{-1}}na).$$

Therefore,

$$\tilde{\beta}(n) = ({}^ga)^{-1}\tilde{\alpha}(n){}^n({}^ga).$$

Therefore, by Remark 2.16,  ${}^g(\alpha, r) \sim {}^g(\beta, s)$ . Thus, the action of  $G$  on  $\text{Der}_c(N, (A, \mu))$  induces an action of  $G$  on  $H^1(N, (A, \mu))$ . Moreover if  $H^1(N, (A, \mu))$  is a group, then  $H^1(N, (A, \mu))$  is a  $G$ -module. It is sufficient to show for every  $m \in N, {}^m(\alpha, r) \sim (\alpha, r)$ . In fact, for every  $n \in N$

$$\begin{aligned} {}^m\alpha({}^{m^{-1}}n) &= {}^m\alpha(m^{-1}nm) = {}^m(\alpha(m^{-1}){}^{m^{-1}}\alpha(n){}^{m^{-1}}n\alpha(m)) = \\ &= {}^m\alpha(m^{-1})\alpha(n){}^n\alpha(m) = \alpha(m)^{-1}\alpha(n){}^n\alpha(m). \end{aligned}$$

Thus,  $H^1(N, (A, \mu))$  is a  $G/N$ -set. Moreover if  $H^1(N, (A, \mu))$  is a group, then  $H^1(N, (A, \mu))$  is a  $G/N$ -module.

(ii) By a similar argument as in (i), for every  $(\alpha, r) \in \text{Der}_c(G, (A, \mu))$

$${}^g\alpha({}^{g^{-1}}n) = \alpha(g)^{-1}\alpha(n){}^n\alpha(g), \forall g \in G, n \in N,$$

whence,  ${}^gN(\alpha \circ \iota, r) \sim (\alpha \circ \iota, r), \forall gN \in G/N$ . □

**Theorem 3.7.** *Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule and  $N$  a normal subgroup of  $G$ . Then, there is an exact sequence*

$$1 \longrightarrow H^1(G/N, (A^N, \mu^N)) \xrightarrow{\text{Inf}^1} H^1(G, (A, \mu)) \xrightarrow{\text{Res}^1} H^1(N, (A, \mu))^{G/N}.$$

*Proof.* The map  $\text{Inf}^1$  is one to one: If  $(\alpha, r), (\beta, s) \in \text{Der}_c(G/N, A^N)$  and  $\text{Inf}^1[(\alpha, r)] = \text{Inf}^1[(\beta, s)]$ , then  $(\alpha\pi, r) \sim (\beta\pi, s)$ . Thus, there is an  $a \in A$  such that  $\beta\pi(g) = a^{-1}\alpha\pi(g){}^ga, \forall g \in G$ . Hence,  $\beta(gN) = a^{-1}\alpha(gN){}^ga, \forall gN \in G/N$ . If  $g \in N$ , then  $\alpha(gN) = \beta(gN) = 1$  and hence,  $a \in A^N$ . This implies that  ${}^ga = ({}^{gN})a, \forall g \in G$ . Consequently,  $(\alpha, r) \sim (\beta, s)$ , i.e.,  $\text{Inf}^1$  is one to one.

Now we show that  $\text{Ker Res}^1 = \text{Im Inf}^1$ . Since  $\text{Res}^1 \text{Inf}^1[(\alpha, r)] = [(\alpha(\pi\iota), r)] = [(\mathbf{1}, 1)]$ , then  $\text{Im Inf}^1 \subset \text{Ker Res}^1$ .

Let  $[(\alpha, r)] \in \text{Ker Res}^1$ . Then, there is an  $a \in A$  such that  $\alpha(n) = a^{-1}{}^na, \forall n \in N$ . Consider  $(\beta, \mu(a)r) \in \text{Der}_c(G, (A, \mu))$  with  $\beta(g) =$

$a\alpha(g)^ga^{-1}$ ,  $\forall g \in G$ . Since  $\beta(n) = 1, \forall n \in N$  then,  $\beta$  induces the continuous crossed homomorphism  $\gamma : G/N \rightarrow A$  via  $\gamma(gN) = \beta(g)$ . Also  $Im\gamma \subset A^N$ , since for all  $n \in N$ ,

$${}^n\gamma(gN) = {}^n\beta(g) = \beta(ng) = \beta(g)^g\beta(g^{-1}ng) = \beta(g) = \gamma(gN).$$

Clearly,  $\mu(a)r \in H^0(N, R)$  and  $(\gamma, \mu(a)r) \in Der_c(G/N, (A^N, \mu^N))$ . Hence,  $Inf^1[(\gamma, \mu(a)r)] = [(\gamma\pi, \mu(a)r)] = [(\beta, \mu(a)r)] = [(\alpha, r)]$ . Consequently,  $KerRes^1 \subset ImInf^1$ .  $\square$

#### 4. COBOUNDARY MAPS AND EXACT SEQUENCE OF COHOMOLOGIES.

In this section we will obtain a seven-term exact sequence of non-abelian cohomologies up to dimension 2.

Suppose that  $1 \rightarrow (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \rightarrow 1$  is an exact sequence of partially crossed topological  $G - R$ -bimodules such that  $\iota$  is an homeomorphic embedding. Thus, we can identify  $A$  with  $\iota(A)$ .

Now we define a coboundary map  $\delta^0 : H^0(G, C) \rightarrow H^1(G, A)$ . Let  $c \in H^0(G, C)$ ,  $b \in B$  with  $\pi(b) = c$ . Then, we define  $\delta^0(c)$  by  $\delta^0(c)(g) = b^{-1}gb, \forall g \in G$ . It is obvious that  $\delta^0(c)$  is a continuous crossed homomorphism. Let  $b' \in B$ ,  $\pi(b') = c$ . Then,  $b' = ba$  for some  $a \in A$ . So,

$$(b')^{-1}gb' = a^{-1}b^{-1}gb^ga = a^{-1}\delta^0(c)(g)^ga.$$

Thus, the crossed homomorphism obtained from  $b'$  is cohomologous in  $A$  to the one obtained from  $b$ , i.e.,  $\delta^0$  is well-defined.

Now, suppose that  $1 \rightarrow (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \rightarrow 1$  is an exact sequence of partially crossed homomorphism such that  $\iota$  is a homeomorphic embedding and in addition  $\pi$  has a continuous section  $s : C \rightarrow B$ .

We construct a coboundary map  $\delta^1 : H^1(G, (C, \lambda)) \rightarrow H^2(G, A)$ . Here,  $H^2(G, A)$  is defined as in [3].

Let  $\alpha \in H^1(G, (C, \lambda))$  and  $s : C \rightarrow B$  be a continuous section for  $\pi$ . Define  $\delta^1$  by  $[(\alpha, r)] \mapsto [\tilde{\alpha}]$ , where  $\tilde{\alpha}(g, h) = s\alpha(g)^g(s\alpha(h))(s\alpha(gh))^{-1}$ . Clearly,  $\tilde{\alpha}$  is a continuous map.

We show that  $\tilde{\alpha}$  is a factor set with values in  $A$ , and independent of the choice of the continuous section  $s$ . Also  $\delta^1$  is well-defined.

Since  $\alpha$  is a crossed homomorphism, we get:

$$\pi(\tilde{\alpha}(g, h)) = \pi(s\alpha(g)^gs\alpha(h)(s\alpha(gh))^{-1}) = \alpha(g)^g\alpha(h)(\alpha(gh))^{-1} = 1.$$

Thus,  $\tilde{\alpha}$  has values in  $A$ .

Next, we show that  $\tilde{\alpha}$  is a factor set, i.e.,

$${}^g\tilde{\alpha}(h, k)\tilde{\alpha}(g, hk) = \tilde{\alpha}(gh, k)\tilde{\alpha}(g, h), \quad \forall g, h, k \in G. \quad (4.1)$$

First we calculate the left hand side of (4.1). For simplicity, take  $b_g = s\alpha(g)$ ,  $\forall g \in G$ . Since  $A \subset \text{Ker}\mu$ , then

$$\begin{aligned} {}^g\tilde{\alpha}(h, k)\tilde{\alpha}(g, hk) &= {}^g(b_h {}^h b_k b_{hk}^{-1})(b_g {}^g b_{hk} b_{ghk}^{-1}) = b_g {}^g (b_h {}^h b_k b_{hk}^{-1}) {}^g b_{hk} b_{ghk}^{-1} \\ &= b_g {}^g (b_h {}^h b_k) {}^g b_{hk} b_{ghk}^{-1} = b_g {}^g b_h {}^g b_k b_{ghk}^{-1}, \end{aligned}$$

On the other hand,

$$\tilde{\alpha}(gh, k)\tilde{\alpha}(g, h) = (b_{gh} {}^{gh} b_k b_{ghk}^{-1})(b_g {}^g b_h b_{gh}^{-1}) = b_g {}^g b_h {}^{gh} b_k b_{ghk}^{-1}.$$

Therefore,  $\tilde{\alpha}$  is a factor set.

Next, we prove that  $[\tilde{\alpha}]$  is independent of the choice of the continuous sections. Suppose that  $s$  and  $u$  are continuous sections for  $\pi$ . Set  $b_g = s\alpha(g)$  and  $b'_g = u\alpha(g)$ . Since  $\pi(b'_g) = \alpha(g) = \pi(b_g)$ , then  $b'_g = b_g a_g$  for some  $a_g \in A$ . Obviously the function  $\kappa : G \rightarrow A$ ,  $g \mapsto a_g$ , is continuous. Thus,

$$\begin{aligned} \bar{\alpha}(g, h) &= b'_g {}^g b'_h b'_{gh} = b_g \kappa(g) {}^g b_h {}^g \kappa(h) (\kappa(gh))^{-1} b_{gh}^{-1} \\ &= (\kappa(g) {}^g \kappa(h) (\kappa(gh))^{-1}) (b_g {}^g b_h b_{gh}^{-1}) = \delta_G^1(\kappa)(g, h) \tilde{\alpha}(g, h), \end{aligned}$$

where  $\delta_G^1(\kappa)(g, h) = {}^g \kappa(h) (\kappa(gh))^{-1} \kappa(g)$ . Consequently,  $\bar{\alpha}$  and  $\tilde{\alpha}$  are cohomologous.

Suppose that  $(\alpha, r)$  and  $(\beta, s)$  are cohomologous in  $\text{Der}_c(G, (C, \lambda))$ . Then, there is  $c \in C$  such that  $\beta(g) = c^{-1}\alpha(g) {}^g c$ ,  $\forall g \in G$ . Let  $s : C \rightarrow A$  be a continuous section for  $\pi$ . Since

$$\pi(s(c^{-1}\alpha(g) {}^g c)) = \pi(s(c)^{-1} s\alpha(g) {}^g s(c)),$$

then, there exists a unique  $\gamma(g) \in \text{ker}\pi = A$  such that

$$\gamma(g)(s(c)^{-1} s\alpha(g) {}^g s(c)) = s(c^{-1}\alpha(g) {}^g c).$$

It is clear that the map  $\gamma : G \rightarrow A$ ,  $g \mapsto \gamma(g)$  is continuous. Therefore,

$$\begin{aligned} \tilde{\beta}(g, h) &= s\beta(g) {}^g s\beta(h) (s\beta(gh))^{-1} \\ &= s(c^{-1}\alpha(g) {}^g c) {}^g s(c^{-1}\alpha(h) {}^h c) (s(c^{-1}\alpha(gh) {}^{gh} c))^{-1} \\ &= \gamma(g)[s(c)^{-1} s\alpha(g) {}^g s(c)] {}^g (\gamma(h)[s(c)^{-1} s\alpha(h) {}^h s(c)]) \\ &\quad \cdot (\gamma(gh)[s(c)^{-1} s\alpha(gh) {}^{gh} s(c)])^{-1} = {}^g \gamma(h) \gamma(gh)^{-1} \gamma(g)[s(c)^{-1} s\alpha(g) {}^g s(c)] \\ &\quad \cdot [s(c)^{-1} s\alpha(h) {}^h s(c)] \cdot [s(c)^{-1} s\alpha(gh) {}^{gh} s(c)]^{-1} \\ &= \delta_G^1(\gamma)(g, h)[s(c)^{-1} s\alpha(g) {}^g s\alpha(h) (a\alpha(gh))^{-1} s(c)] \end{aligned}$$

$$= \delta_G^1(\gamma)(g, h)[s(c)^{-1}\delta^1(\alpha)(g, h)s(c)] = \delta_G^1(\gamma)(g, h)\tilde{\alpha}(g, h).$$

The last equality is obtained from the fact that  $\tilde{\alpha}(g, h) \in \text{Ker}\mu$  and  $s(c) \in B$ . Now, note that  $\tilde{\alpha}$  is cohomologous to  $\tilde{\beta}$ , when  $(\alpha, r)$  is cohomologous to  $(\beta, s)$ . Thus,  $\delta^1$  is well-defined.

Recall that a short sequence

$$1 \rightarrow (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \rightarrow 1$$

of partially crossed topological  $G$ - $R$ -bimodules is exact, if the following diagram is commutative and the top row is exact.

$$\begin{array}{ccccccc} 1 & \longrightarrow & A & \xrightarrow{\iota} & B & \xrightarrow{\pi} & C \longrightarrow 1 \\ & & & & \downarrow \mu & & \uparrow \lambda \\ & & & & R & & \end{array}$$

The above exact sequence is called a proper extension with continuous sections, if  $1 \rightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \rightarrow 1$  is a proper extension and  $\pi$  has a continuous section.

**Theorem 4.1.** *Let  $1 \rightarrow (A, \mathbf{1}) \xrightarrow{\iota} (B, \mu) \xrightarrow{\pi} (C, \lambda) \rightarrow 1$  be a proper extension of partially crossed topological  $G$ - $R$ -bimodules with continuous sections. Then, the following sequence is exact.*

$$\begin{aligned} 0 \rightarrow H^0(G, A) &\xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \xrightarrow{\iota^1} \\ H^1(G, (B, \mu)) &\xrightarrow{\pi^1} H^1(G, (C, \lambda)) \xrightarrow{\delta^1} H^2(G, A) \end{aligned}$$

*Proof.* We may assume that  $\iota$  is the inclusion map.

1. By [8, Theorem 4.1], the sequence

$$0 \rightarrow H^0(G, A) \xrightarrow{\iota^0} H^0(G, B) \xrightarrow{\pi^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A)$$

is exact.

2. Exactness at  $H^1(G, A)$ : Let  $c \in H^0(G, C)$ . Then, there is  $b \in B$  such that  $\pi(b) = c$ . So,

$$\iota^1 \delta^0(c)(g) = \iota(\delta^0(c)(g)) = \iota(b^{-1}gb) = b^{-1}gb.$$

Consequently,  $\iota^1 \delta^0(c) \sim \mathbf{1}$ . Conversely, let  $[\alpha] \in \text{Ker}\iota^1$ . Then, there is  $b \in B$  such that  $\alpha(g) = b^{-1}gb, \forall g \in G$ . So,  $\pi(b^{-1}gb) = 1, \forall g \in G$ . Take  $c = \pi(b)$ . Hence,  $c \in H^0(G, C)$ . Thus,  $\delta^0(c) \sim \alpha$ .

3. Exactness at  $H^1(G, (B, \mu))$ : Since  $\pi^1 \iota^1([\alpha]) = \pi^1([\alpha, 1]) = [(\pi \circ \alpha, 1)] = [(\mathbf{1}, 1)] = 1$ , then,  $\text{Im}\iota^1 \subset \text{Ker}\pi^1$ . Conversely, let  $[(\beta, r)] \in \text{ker}\pi^1$ . Then, there is  $c \in C$  such that  $\pi\beta(g) = c^{-1}gc$ , for all  $g \in G$  and  $r = \lambda(c)^{-1}z$ , for some  $z \in H^0(G, R)$ . Let  $b \in B$  and  $c = \pi(b)$ . Therefore,  $\pi(\beta(g)) = \pi(b^{-1}gb), \forall g \in G$ . On the other hand, the map

$\tau_b : A \rightarrow A$ ,  $a \mapsto b^{-1}ab$ , is a topological isomorphism, because  $A$  is a normal subgroup of  $B$ . So, for every  $g \in G$  there is a unique element  $a_g \in G$  such that  $\beta(g) = (b^{-1}a_gb)(b^{-1}gb)$ . Thus,  $\beta(g) = b^{-1}a_g b$ ,  $\forall g \in G$ . Hence,  $a_g = b\beta(g)g^{-1}$ ,  $\forall g \in G$ . Obviously, the map  $\alpha : G \rightarrow A$ ,  $g \mapsto a_g$ , is a continuous crossed homomorphism, and  $\iota^1([\alpha]) = [(\alpha, 1)] = [(\alpha, z)] = [(\alpha, \lambda(c)r)] = [(\alpha, \mu(b)r)] = [(\beta, r)]$ .

4. Exactness at  $H^1(G, (C, \lambda))$ : Let  $[(\beta, r)] \in H^1(G, B)$  and  $s$  be a continuous section for  $\pi$ . Then  $\delta^1 \circ \pi^1([( \beta, r)]) = [\widetilde{\pi\beta}]$ . There is a continuous map  $z : G \rightarrow A$  such that  $s\pi\beta(g) = \beta(g)z(g)$ . Thus,

$$\begin{aligned} \widetilde{\pi\beta}(g, h) &= s(\pi\beta(g))^g s(\pi\beta(h))(s(\pi\beta(gh)))^{-1} = \\ &= \beta(g)^g \beta(h) \beta(gh)^{-1} \delta_G^1(z)(g, h) = \delta_G^1(z)(g, h). \end{aligned}$$

So,  $Im\pi^1 \subset Ker\delta^1$ . Conversely, let  $[(\gamma, r)] \in ker\delta^1$ . Then, there is a continuous function  $\alpha : G \rightarrow A$  such that  $\tilde{\gamma} = \delta_G^1(\alpha)$ , where  $[\tilde{\gamma}] = \delta^1([( \gamma, r)])$ . Thus,

$$\tilde{\gamma}(g, h) = s\gamma(g)^g s\gamma(h)(s\gamma(gh))^{-1} = {}^g\alpha(h)\alpha(gh)^{-1}\alpha(g), \forall g, h \in G.$$

Assume  $\beta(g) = s\gamma(g)\alpha(g)^{-1}$ ,  $\forall g \in G$ . Since  $A \subset Ker\mu$ , then  $\beta$  is a continuous crossed homomorphism from  $G$  to  $B$ , and  $\pi\beta = \gamma$ . Also  $(\beta, r) \in Der_c(G, (B, \mu))$  because  $\mu\beta(g) = \mu(s\gamma(g)\alpha(g)^{-1}) = \mu(s\gamma(g)) = \lambda\gamma(g) = r^g r^{-1}$ . Hence,  $\pi^1([( \beta, r)]) = [(\gamma, r)]$ . This completes the proof.  $\square$

## 5. PRINCIPAL HOMOGENEOUS SPACES OVER $(A, \mu)$ - A NEW DEFINITION OF $H^1(G, (A, \mu))$

Serre [9] showed that if  $A$  is a topological  $G$ -module in which  $A$  is discrete and  $G$  a profinite group then there is a bijection between the set,  $P(A)$ , of all classes of principal homogeneous spaces over  $A$  and  $H^1(G, A)$ . Similarly, Inassaridze [4] (algebraically) defined the  $G$ -torsor over a crossed  $G$ -module  $(A, \mu)$  and showed that there is a natural isomorphism between the group,  $E(G, A)$ , of all classes of  $G$ -torsors over  $(A, \mu)$  and the Guin's first non-abelian cohomology group  $H^1(G, (A, \mu))$  (see [4, Theorem 4.2]). We will show that the first non-abelian cohomology of topological groups is closely related with principle homogeneous spaces.

**Definition 5.1.** A principal homogeneous space (or topological torsor) over a partially crossed topological  $G - R$ -bimodule  $(A, \mu)$  is a pair  $(P, f)$  consisting of a  $G$ -space  $P$ , on which  $A$  acts on the right (in a manner compatible with  $G$ ) so that for any  $p \in P$  the natural map  $e_p : A \rightarrow P$ ,  $a \mapsto pa$  is a homeomorphism, and  $f$  is a  $G$ -map from  $P$  to  $R$  such that  $f(pa) = \mu^{-1}(a)f(p)$  for any  $p \in P$ ,  $a \in A$ .



**Definition 5.2.** It is said that principal homogeneous spaces  $(P, f)$  and  $(Q, g)$  over a partially crossed topological  $G - R$ -bimodule  $(A, \mu)$  are isomorphic if there is a homomorphism  $\nu : P \rightarrow Q$  compatible with the actions of  $G$  and  $A$  such that  $f(p) = g \circ \nu(p) \bmod H^0(G, R)$  for any  $p \in P$ .

Obviously, the isomorphism in Definition 5.2 is an equivalence relation in the set of all principle homogenous spaces over  $(A, \mu)$ . We denote by  $\mathcal{P}(A, \mu)$  the set of all classes of principal homogeneous spaces over  $(A, \mu)$ . Suppose that  $A$  acts on the right on itself by translations. Obviously,  $(A, \mu^{-1})$  is a principal homogeneous space over  $(A, \mu)$ . We call it trivial topological torsor over  $(A, \mu)$ . Thus,  $\mathcal{P}(A, \mu) \neq \emptyset$ .

**Theorem 5.3.** Let  $(A, \mu)$  be a partially crossed topological  $G - R$ -bimodule. There is a bijection between  $\mathcal{P}(A, \mu)$  and  $H^1(G, (A, \mu))$ .

*Proof.* If  $[(P, f)] \in \mathcal{P}(A, \mu)$ , we choose a point  $p \in P$ . If  $g \in G$ , one has  ${}^g p \in P$ , therefore there exists a unique  $\alpha_p^P(g) \in A$  such that  ${}^g p = p\alpha_p^P(g)$ . One can see that  $g \mapsto \alpha_p^P(g)$  is a continuous crossed homomorphism and  $\mu\alpha_p^P(g) = f(p)f(p\alpha_p^P(g))^{-1} = f(p)f({}^g p)^{-1} = f(p){}^g f(p)^{-1}$ . Thus,  $(\alpha_p^P, f(p)) \in \text{Der}_c(G, (A, \mu))$ . On the one hand, substituting  $pa$  for  $p$  changes this crossed homomorphism into  $g \mapsto a^{-1}\alpha_p^P(g){}^g a$ , since  ${}^g(pa) = {}^g p {}^g a = p\alpha_p^P(g){}^g a = (pa)a^{-1}\alpha_p^P(g){}^g a$ . Also,  $f(pa) = \mu(a)^{-1}f(p)$ . Therefore,  $(\alpha_q^P, f(q)) \sim (\alpha_p^P, f(p))$  for any  $q \in P$ . On the other hand, let  $(P', f')$  be isomorphic to  $(P, f)$ . Thus, there is a homeomorphism  $\nu : P \rightarrow P'$  with properties in Definition 5.2. Let  $p' \in P'$  and  $\nu(p) = p'$ . Then  ${}^g p' = {}^g \nu(p) = \nu({}^g p) = \nu(p\alpha_p^P(g)) = \nu(p)\alpha_p^P(g)$ . So,  $\alpha_{p'}^{P'} = \alpha_p^P$ . Therefore by Remark 2.16,  $(\alpha_{p'}^{P'}, f'(p')) \sim (\alpha_p^P, f(p))$ . One may thus define  $\lambda : \mathcal{P}(A, \mu) \rightarrow H^1(G, (A, \mu))$  via  $[(P, f)] \mapsto [(\alpha_p^P, f(p))]$ .

Vise versa, one defines  $\gamma : H^1(G, (A, \mu)) \rightarrow \mathcal{P}(A, \mu)$  as follows: If  $(\alpha, r) \in \text{Der}_c(G, (A, \mu))$ , denote by  $P_\alpha$  the group  $A$  on which  $G$  acts by the following *twisted formula*:

$$ga = \alpha(g){}^g a.$$

Let now  $A$  act on the right on  $P_\alpha$  by translations. Define  $f_r : P_\alpha \rightarrow R$  by  $f_r(a) = \mu^{-1}(a)r$ . We have

$$f_r(ga) = \mu^{-1}(\alpha(g){}^g a)r = {}^g \mu(a)^{-1} \mu^{-1}(\alpha(g))r = {}^g (\mu(a)^{-1}r) = {}^g f_r(a)$$

for any  $g \in G$ . Thus,  $f_r$  is a  $G$ -map. In addition, for any  $a, b \in A$ ,

$$f_r(ab) = \mu(ab)^{-1}r = \mu(b)^{-1}(\mu(a)^{-1}r) = \mu(b)^{-1}f_r(a).$$

Therefore,  $(P_\alpha, f_r)$  is a principal homogeneous space over  $(A, \mu)$ .

If  $(\alpha, r) \sim (\beta, s)$ , then there is  $a \in A$  such that  $\beta(g) = a^{-1}\alpha(g)^ga$  for any  $g \in G$ , and  $s = \mu(a)^{-1}rt$  for some  $t \in H^0(G, R)$ . Define  $\nu : P_\alpha \rightarrow P_\beta$  by  $p \mapsto a^{-1}p$ . For every  $g \in G, p \in P_\alpha$ , then

$$\nu(gp) = \nu(\alpha(g)^gp) = a^{-1}\alpha(g)^gp = \beta(g)^ga^{-1}p = \beta(g)^g(a^{-1}p) = g\nu(p).$$

Thus,  $\nu$  is a  $G$ -map. Obviously,  $\nu$  is compatible with the action  $A$  on  $P_\alpha$  and  $P_\beta$ .

For  $p \in P_\alpha$ , we get

$$f_s(\nu(p)) = f_s(a^{-1}p) = \mu(p^{-1}a)s = \mu^{-1}(p)\mu(a)\mu(a)^{-1}rt = \mu^{-1}(p)rt.$$

Therefore,  $(P_\alpha, f_r)$  is isomorphic to  $(P_\beta, f_s)$ . Consequently, one can define  $\gamma$  by  $\gamma([\alpha, r]) = [(P_\alpha, f_r)]$ .

We will show that  $\gamma\lambda = Id_{P(A, \mu)}$ . Let  $(Q, g)$  be a principle homogeneous space over  $(A, \mu)$ . Fix  $q \in Q$ . We define  $\nu : Q \rightarrow P_{\alpha_q}$  by  $p \mapsto e_q^{-1}(p)$ . Obviously  $\nu$  is a homeomorphism. For any  $h \in G$ ,

$$\nu(hp) = e_q^{-1}(hp) = e_q^{-1}(h(qe_q^{-1}(p))) = e_q^{-1}(h^h q^h e_q^{-1}(p)) = {}^h e_q^{-1}(p) = {}^h \nu(p).$$

i.e.,  $\nu$  is a  $G$ -map. Also for any  $a \in A$ ,

$$\nu(pa) = e_q^{-1}(pa) = e_q^{-1}(qe_q^{-1}(p)a) = e_q^{-1}(p)a.$$

In addition for any  $p \in P$ ,

$$f_{g(q)} \circ \nu(p) = \mu^{-1}(\nu(p))g(q) = g(q\nu(p)) = g(qe_q^{-1}(p)) = g(p).$$

This implies that  $\gamma\lambda = Id_{P(A, \mu)}$ .

Conversely, let  $(\beta, s) \in Der_c(G, (A, \mu))$ . For every  $p \in P_\beta$ ,  $f_s(p) = \mu^{-1}(p)s$  and also for every  $g \in G$ ,  $p\alpha_p^{P_\beta}(g) = gp = \alpha(g)^gp$ . Thus,  $(\beta, s) \sim (\alpha_p^{P_\beta}, f_s(p))$ . This shows that  $\lambda\gamma = Id_{H^1(G, (A, \mu))}$ .  $\square$

*Remark 5.4.* In Definition 5.1, we may consider the  $G$ -map  $f : P \rightarrow R$  as a continuous one.

*Notice 5.5.* If  $Der_c(G, (A, \mu))$  induces naturally a group structure on  $H^1(G, (A, \mu))$ , then we will define a group product on  $\mathcal{P}(A, \mu)$ . Let  $[(P_1, f_1)], [(P_2, f_2)] \in \mathcal{P}(A, \mu)$ . Fix  $p_1 \in P_1$  and  $p_2 \in P_2$ . Set  $P = A$  and let  $A$  act on the right on  $P$  by translations. Consider a new action of  $G$  on  $P$  by the formula:

$$ga = {}^{f_1(p_1)}\alpha_{p_2}^{P_2}(g)\alpha_{p_1}^{P_1}(g)^ga$$

Define the map  $f : P \rightarrow R$  by  $f(a) = \mu^{-1}(a)f_1(p_1)f_2(p_2)$ . We will show that  $(P, f)$  is a principle homogeneous space over  $(A, \mu)$ . Using  $\lambda, \gamma$  in the proof of Theorem 2.23, we consider the classes  $[(\alpha_{p_1}^{P_1}, f_1(p_1))]$

and  $[(\alpha_{p_2}^{P_2}, f_2(p_2))]$  corresponding to the classes  $[(P_1, f_1)]$  and  $[(P_2, f_2)]$ , respectively. Since  $H^1(G, (A, \mu))$  is group, then

$$[(\alpha_{p_1}^{P_1}, f_1(p_1))][(\alpha_{p_2}^{P_2}, f_2(p_2))] = [({}^{f_1(p_1)}\alpha_{p_2}^{P_2}\alpha_{p_1}^{P_1}, f_1(p_1)f_2(p_2))].$$

Obviously  $\gamma([({}^{f_1(p_1)}\alpha_{p_2}^{P_2}\alpha_{p_1}^{P_1}, f_1(p_1)f_2(p_2))]) = [(P, f)]$ . Thus, one can define a group product  $\circ$  as follows:

$$[(P_1, f_1)] \circ [(P_2, f_2)] = [(P, f)].$$

**Corollary 5.6.** *Let  $(A, \mu)$  be a partially crossed topological  $G$ -module. Then,  $(\mathcal{P}(A, \mu), \circ)$  is a group.*

*Proof.* It is clear that by Theorem 2.23,  $H^1(G, (A, \mu))$  is a group and Notice 5.5 implies that  $\mathcal{P}(A, \mu)$  is a group by the action  $\circ$ .  $\square$

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### H.E. Koshkoshi

Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran.

Email: [h.e.koshkoshi@guilan.ac.ir](mailto:h.e.koshkoshi@guilan.ac.ir)

### H. Sahleh

Department of Mathematics, Faculty of Mathematical Sciences, University of Guilan, P.O.Box 1914, Rasht, Iran.

Email: [sahleh@guilan.ac.ir](mailto:sahleh@guilan.ac.ir)